Spatially Distributed Model of Interacting Surface and Groundwater Storages

Thomas E. Croley II[1]

[1]Great Lakes Environmental Research Laboratory, National Oceanic and Atmospheric Administration, 2205 Commonwealth Blvd., Ann Arbor, Michigan 48105-2945; PH (734) 741-2238; FAX (734) 741-2055; email: Tom.Croley@noaa.gov

Abstract

NOAA's Great Lakes Environmental Research Laboratory (GLERL) developed their Large Basin Runoff Model (LBRM) as a serial and parallel cascade of linear reservoirs representing moisture storages within a watershed. Each reservoir represents a moisture storage: surface, upper soil zone, lower soil zone, and groundwater zone. GLERL adapted the LBRM from its lumped-parameter definition for an entire watershed to a two dimensional representation of the flow cells comprising the watershed. This involved changes to the model structure to apply it to the micro scale as well as organization of watershed cells and an implementation of spatial flow routing.

GLERL modified the LBRM continuity equations to allow upstream inflow when the model is applied to a single cell within a watershed and found the modifications in terms of corrector equations to be applied to the original solution. They began by considering flows between adjacent cells' surface storages while keeping the upper soil zone, lower soil zone, and groundwater zones in each cell independent. Thus each cell's upper soil zone, lower soil zone, and groundwater zone connected only to that cell's surface zone and not to any other cell, but the surface zones connected between adjacent cells. GLERL further modified the model to allow subsurface routing between cells of surface runoff (from the upper soil zone), interflow (from the lower soil zone), and groundwater flows (from the groundwater zone). Now surface and subsurface flows interact both with each other and with adjacent-cell surface and subsurface storages. This involved adding additional flows out of the various subsurface storages in a watershed cell and additional flows (from upstream watershed cells' subsurface storages) into the storages

LBRM Structural Modification

GLERL developed a large-scale operational model in the 1980s for estimating rainfall/runoff relationships on the 121 large watersheds surrounding the Laurentian Great Lakes. It is physically based to provide good representations of hydrologic processes and to ensure that results are tractable and explainable. It is used here in application to individual sub areas (cells) within a watershed by modifying its structure to accept upstream flows. The unmodified mass balance schematic is shown in Figure 1. Daily precipitation, temperature, and insolation (the latter available from meteorological summaries as a function of location) may be used to determine snow pack accumulations and net supply, s. The net supply is divided into surface runoff, $s\frac{U}{C}$, and infiltration to the upper soil zone, $s - s\frac{U}{C}$, in relation to the upper soil zone moisture content, U, and the fraction it represents of the upper soil zone capacity, C. Percolation to the lower soil zone, $\alpha_p U$, and evapotranspiration, $\beta_u e_p U$, are taken as outflows from a linear reservoir (flow is proportional to storage). Likewise, interflow from the lower soil zone to the surface, $\alpha_i L$, evapotranspiration, $\beta_\ell e_p L$, and deep percolation to the groundwater zone, $\alpha_d L$, are linearly proportional to the lower soil zone moisture content, L. Groundwater flow, $\alpha_g G$, and evapotranspiration from the groundwater zone, $\beta_g e_p G$, are linearly proportional to the groundwater zone moisture content, G. Finally, basin outflow, $\alpha_s S$, and evaporation from the surface storage, $\beta_s e_p S$, depend on its content, S. Additionally, evaporation and evapotranspiration are dependent on potential evapotranspiration, e_p, as determined by joint consideration of the available moisture and the heat balance over the watershed. The "alpha" coefficients (α) represent linear reservoir proportionality factors and the "beta" coefficients (β) represent partial linear reservoir coefficients associated with evapotranspiration.

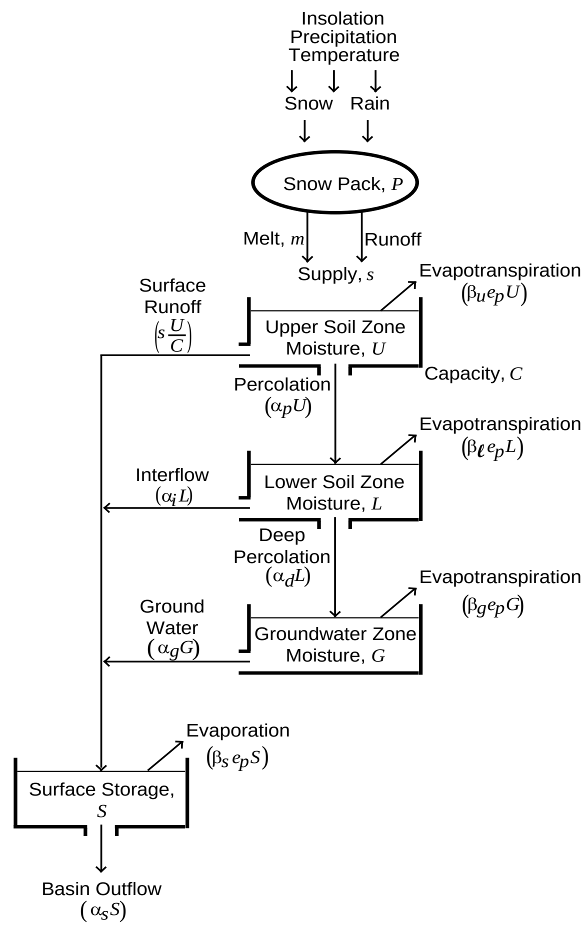

Figure 1. LBRM Tank Cascade Schematic.

Mass conservation equations (Croley 2002) are repeated here for convenience as differential equations with respect to time t.

$$\frac{d}{dt}U = s\left(1-\frac{U}{C}\right) - \alpha_p U - \beta_u e_p U \tag{1}$$

$$\frac{d}{dt}L = \alpha_p U - \alpha_i L - \alpha_d L - \beta_\ell e_p L \tag{2}$$

$$\frac{d}{dt}G = \alpha_d L - \alpha_g G - \beta_g e_p G \tag{3}$$

$$\frac{d}{dt}S = s\frac{U}{C} + \alpha_i L + \alpha_g G - \alpha_s S - \beta_s e_p S \tag{4}$$

Equations (1)—(4) can be expressed in the general form:

$$dZ + \left(\sum\alpha\right) Z\, dt = f\left(t\right) dt \tag{5}$$

where Z = storage, $\left(\sum\alpha\right)$ = sum of linear reservoir constants for all outflows, and $f\left(t\right)$ = sum of time-dependent inflows. Standard procedures (Rainville 1964) yield:

$$Z_t = e^{-\left(\sum\alpha\right)t}\left[Z_0 + \int_0^t f\left(u\right) e^{\left(\sum\alpha\right)u} du\right] \tag{6}$$

where the subscript is time. In solving (1)—(4) for some time increment $\left(0,t\right)$, we generally take net supply and potential evapotranspiration as uniform over the increment. Storage values at the end of a time increment are computed from values at the beginning. In the analytical solution, results from one storage zone are used in other zones where their outputs appear as inputs. There are several different solutions, depending upon the relative magnitudes of all coefficients in (1)—(4). Croley (2002) solved the equations, yielding storages at the end of a time increment (U_t, L_t, G_t, and S_t) as functions of the inputs, parameters, and beginning-of-time-increment storages (storages at the end of the previous time increment: U_0, L_0, G_0, and S_0). Since the variables s and e_p change from one time increment to another, then the appropriate analytical result, as well as its solution, varies with time. Mathematical continuity between solutions is preserved however. These results are summarized elsewhere (Croley 1982). The solution for surface storage from (4) via (5) and (6) is.

$$S_t = e^{-\left(\alpha_s + \beta_s e_p\right)t}\left[S_0 + \int_0^t \left(s\frac{U}{C} + \alpha_i L + \alpha_g G\right) e^{\left(\alpha_s + \beta_s e_p\right)u} du\right] \tag{7}$$

where S_t is the surface storage at the end of the time increment $\left(0,t\right)$. Croley (2002) shows that, in all cases, flow volumes are determined directly since outflow volumes are related by their ratio of linear reservoir coefficients. In particular, the volume of basin outflow (from the surface storage) over the time increment $\left(0,t\right)$, is

$$V_s = \left(V_r + V_i + V_g + S_0 - S_t\right)\frac{\alpha_s}{\alpha_s + \beta_s e_p} \tag{8}$$

where V_s = basin outflow volume from surface storage and V_r = surface runoff volume, V_i = interflow volume, and V_g = groundwater flow volume, all into the surface storage, over increment $(0,t)$.

Upstream Surface Flow

An amended mass balance schematic for a watershed cell is shown in Figure 2, where an upstream surface flow is added (from an upstream cell). The mass balance is the same employed by Croley (2002) except for adding the upstream surface flow, h. By using the above nomenclature for the case with no upstream cell flow $(h=0)$ and a "prime" notation for the case with upstream cell flow, (4) becomes:

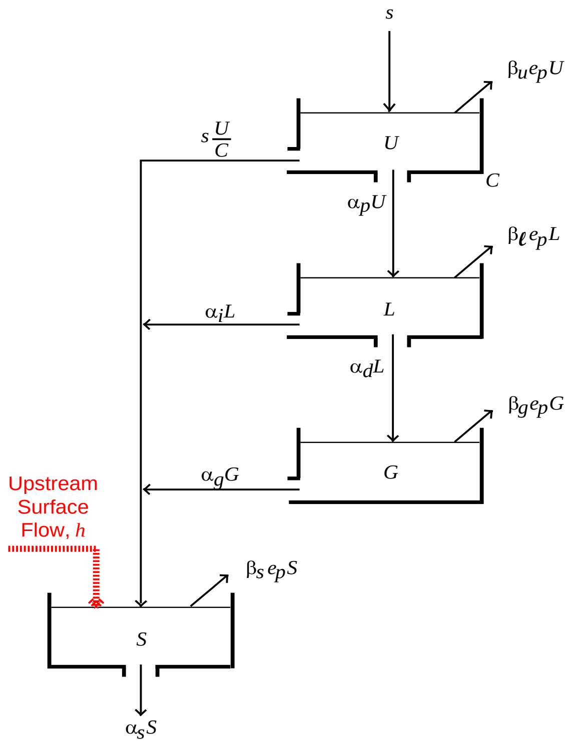

Figure 2. LBRM with Surface Inflow.

$$\frac{d}{dt}S' = s\frac{U}{C} + \alpha_i L + \alpha_g G - \alpha_s S' - \beta_s e_p S' + h \tag{9}$$

Solving (9) via (5) and (6) yields

$$S_t' = e^{-(\alpha_s+\beta_s e_p)t}\left[S_0' + \int_0^t \left(s\frac{U}{C} + \alpha_i L + \alpha_g G + h\right) e^{(\alpha_s+\beta_s e_p)u}\,du\right] \tag{10}$$

If we approximate h as constant over the time increment $(0,t)$, (10) becomes

$$S_t' = e^{-(\alpha_s+\beta_s e_p)t}\left[S_0' + \int_0^t \left(s\frac{U}{C} + \alpha_i L + \alpha_g G\right) e^{(\alpha_s+\beta_s e_p)u}\,du\right] + e^{-(\alpha_s+\beta_s e_p)t} h \int_0^t e^{(\alpha_s+\beta_s e_p)u}\,du \tag{11}$$

Starting from the same initial storage, S_0' = S_0 at t = 0; from (7), (11) becomes

$$S_t' = S_t + h\frac{\left(1 - e^{-(\alpha_s+\beta_s e_p)t}\right)}{\alpha_s + \beta_s e_p} \tag{12}$$

Likewise, (8) becomes

$$V_s' = \left(V_r + V_i + V_g + S_0' - S_t' + h\,t\right)\frac{\alpha_s}{\alpha_s + \beta_s e_p} = V_s + h\left(t - \frac{\left(1 - e^{-\left(\alpha_s + \beta_s e_p\right)t}\right)}{\alpha_s + \beta_s e_p}\right)\frac{\alpha_s}{\alpha_s + \beta_s e_p} \quad (13)$$

Therefore, the output of the LBRM, applied to a single cell with no inflow from an upstream cell, (S_t and V_s) can be corrected each time increment with (12) and (13) to reflect the presence of an inflow, h, from an upstream cell (S_t' and V_s'). The beginning storage in the following time increment is set equal to the ending storage for the present time increment, S_t'. The outflow volume from the cell, V_s', determines the inflow to the next downstream cell; again approximating it as constant over the time interval, it is determined by dividing by the time interval.

Upstream Groundwater Flow

The LBRM similarly can be further expanded to include upstream groundwater flows. First we must allow an additional flow out of the groundwater storage (to be passed to the downstream cell as an upstream groundwater flow). Since the groundwater storage is represented as a linear reservoir, this additional flow will be $\alpha_w G$ where α_w is the linear reservoir coefficient governing groundwater flows directly to downstream cell groundwater storages. Then we must allow the upstream flow into the groundwater storage, g; see Figure 3. By again using the preceding nomenclature for the case with no upstream cell flow to the groundwater storage (g = 0), the general solution for groundwater storage from (3) via (5) and (6) is

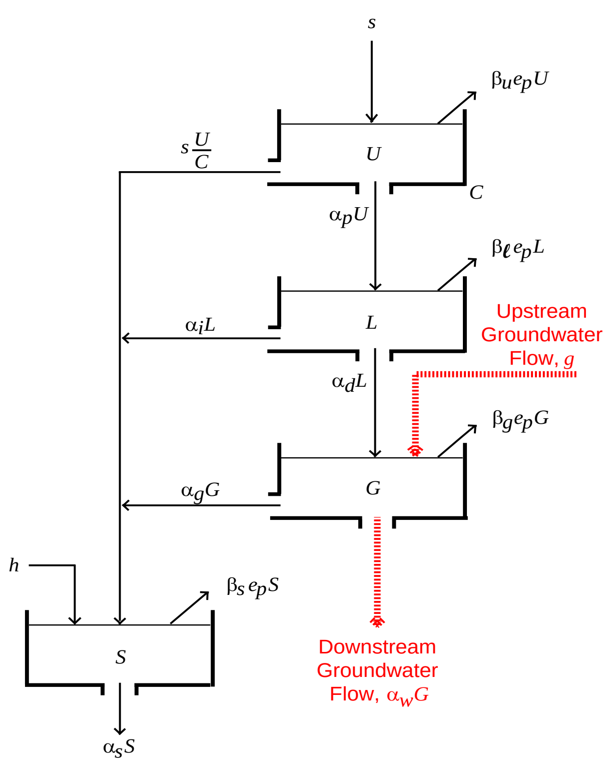

Figure 3. LBRM with Groundwater Inflow.

$$G_{t,\alpha_g} = e^{-\left(\alpha_g + \beta_g e_p\right)t}\left(G_0 + \int_0^t \alpha_d L\, e^{\left(\alpha_g + \beta_g e_p\right)u}\,du\right) \quad (14)$$

$$V_{g,\alpha_g} = \left(V_d + G_0 - G_{t,\alpha_g}\right)\frac{\alpha_g}{\alpha_g + \beta_g e_p} \quad (15)$$

where G_{t,α_g} and V_{g,α_g} are, respectively, storage at the end of time increment $(0,t)$ and groundwater flow volume into surface storage, both written as a function of α_g for convenience later. Considering now $g \neq 0$, (3) and its solution become:

$$\frac{d}{dt}G' = \alpha_d L - \alpha_g G' - \beta_g e_p G' - \alpha_w G' + g \tag{16}$$

$$G'_t = e^{-(\alpha_g+\alpha_w+\beta_g e_p)t}\left[G'_0 + \int_0^t (\alpha_d L + g)\, e^{(\alpha_g+\alpha_w+\beta_g e_p)u} du\right] \tag{17}$$

If we approximate g as constant over the time interval $(0,t)$:

$$\begin{aligned} G'_t &= e^{-(\alpha_g+\alpha_w+\beta_g e_p)t}\left(G'_0 + \int_0^t \alpha_d L\, e^{(\alpha_g+\alpha_w+\beta_g e_p)u} du\right) \\ &\quad + e^{-(\alpha_g+\alpha_w+\beta_g e_p)t} g \int_0^t e^{(\alpha_g+\alpha_w+\beta_g e_p)u} du \end{aligned} \tag{18}$$

Now, for $G'_0 = G_0$ at $t = 0$, we have

$$G'_t = G_{t,\alpha_g+\alpha_w} + g\,\frac{1-e^{-(\alpha_g+\alpha_w+\beta_g e_p)t}}{\alpha_g+\alpha_w+\beta_g e_p} \tag{19}$$

$$\begin{aligned} V'_g + V'_w = V'_{g+w} &= (V_d + G'_0 - G'_t + g\,t)\frac{\alpha_g+\alpha_w}{\alpha_g+\alpha_w+\beta_g e_p} \\ &= V_{g,\alpha_g+\alpha_w} + g\left(t - \frac{1-e^{-(\alpha_g+\alpha_w+\beta_g e_p)t}}{\alpha_g+\alpha_w+\beta_g e_p}\right)\frac{\alpha_g+\alpha_w}{\alpha_g+\alpha_w+\beta_g e_p} \end{aligned} \tag{20}$$

where the subterranean outflow volume, V'_w, from the groundwater storage $\left(V'_w = \frac{\alpha_w}{\alpha_g+\alpha_w}V'_{g+w}\right)$ determines the groundwater inflow to the next downstream cell. Again, approximating it as constant over the time interval, it is determined by dividing by the length of the time interval. Therefore, existing computer code in the LBRM encoding can be applied for upstream ground-water flow, g, into the groundwater storage by substituting $\alpha_g + \alpha_w$ for α_g in (14) and (15) and correcting each time increment with (19) and (20). The beginning storage in the following time increment is set equal to the ending storage for the present time increment, G'_t. Furthermore, the outflow volume from the cell's groundwater storage, V'_g, would be used (as V_g) in (8) to compute the basin outflow volume, V_s, which is then used in (13) to compute the corrected basin outflow volume, V'_s.

Other Storage Upstream Flows

The LBRM similarly can be further expanded to include other flows into the upper and lower soil zone storages from respective upstream cells' storages; see Figure 4. Again, first we must allow an additional flow out of each storage (to be passed to the downstream cell's respective storage as an upstream flow). Since the storages are represented as linear reservoirs, these additional flows will be $\alpha_u U$ and $\alpha_\ell L$, respectively from the upper and lower soil zone moisture storages, where α_u and α_ℓ are the linear reservoir coefficients. Then we must allow upstream flows, u and ℓ, into these storages, respectively. By again using the original nomenclature for the case with no upstream cell flow to the upper or lower soil zone moisture storages, the general solution for each can be found in terms of function definitions already coded and corrector equations as was the case for the groundwater storage. They are derived and defined similarly; for the upper soil zone,

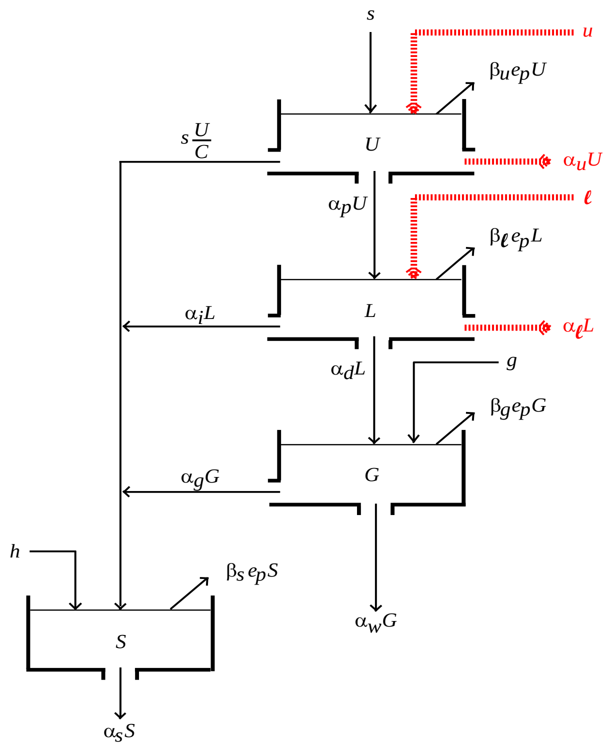

Figure 4. LBRM with *U* and *L* Zone Inflows.

$$U_{t,\frac{s}{C}} = e^{-\left(\frac{s}{C}+\alpha_p+\beta_u e_p\right)t}\left(U_0+\int_0^t s\, e^{\left(\frac{s}{C}+\alpha_p+\beta_u e_p\right)u}\,du\right) \qquad (21)$$

$$V_{r,\frac{s}{C}} = \left(V_x+U_0-U_{t,\frac{s}{C}}\right)\frac{\frac{s}{C}}{\frac{s}{C}+\alpha_p+\beta_u e_p} \qquad (22)$$

$$U_t' = U_{t,\frac{s}{C}+\alpha_u} + u\,\frac{1-e^{-\left(\frac{s}{C}+\alpha_u+\alpha_p+\beta_u e_p\right)t}}{\frac{s}{C}+\alpha_u+\alpha_p+\beta_u e_p} \qquad (23)$$

$$V_r' + V_u' = V_{r+u}' = V_{r,\frac{s}{C}+\alpha_u} + u\left(t-\frac{1-e^{-\left(\frac{s}{C}+\alpha_u+\alpha_p+\beta_u e_p\right)t}}{\frac{s}{C}+\alpha_u+\alpha_p+\beta_u e_p}\right)\frac{\frac{s}{C}+\alpha_u}{\frac{s}{C}+\alpha_u+\alpha_p+\beta_u e_p} \qquad (24)$$

where V_x = volume of supply to the upper soil zone $(= st)$ and the subterranean outflow volume, $V_u' = \dfrac{\alpha_u}{\dfrac{s}{C} + \alpha_u} V_{r+u}'$, from the upper soil zone storage determines the upper soil zone inflow to the next downstream cell. Again, approximating it as constant over the time interval, it is determined by dividing by the length of the time interval. Existing computer code in the LBRM encoding can be applied for upstream flow into the upper soil zone, u, by substituting $\dfrac{s}{C} + \alpha_u$ for $\dfrac{s}{C}$ in (21) and (22) and correcting each time increment with (23) and (24). The beginning storage in the following time increment is set equal to the ending storage for the present time increment, U_t'. Furthermore, the outflow volume from the cell's upper soil zone storage, V_r', would be used (as V_r) in (8) to compute the basin outflow volume, V_s, which is then used in (13) to compute the corrected basin outflow volume, V_s'.

For the lower soil zone,

$$L_{t,\alpha_i} = e^{-\left(\alpha_i + \alpha_d + \beta_\ell e_p\right)t} \left(L_0 + \int_0^t \alpha_p U \; e^{\left(\alpha_i + \alpha_d + \beta_\ell e_p\right)u} du \right) \tag{25}$$

$$V_{i,\alpha_i} = \left(V_p + L_0 - L_{t,\alpha_i} \right) \frac{\alpha_i}{\alpha_i + \alpha_d + \beta_\ell e_p} \tag{26}$$

$$L_t' = L_{t,\alpha_i + \alpha_\ell} + \ell \frac{1 - e^{-\left(\alpha_i + \alpha_\ell + \alpha_d + \beta_\ell e_p\right)t}}{\alpha_i + \alpha_\ell + \alpha_d + \beta_\ell e_p} \tag{27}$$

$$V_i' + V_\ell' = V_{i+\ell}' = V_{i,\alpha_i + \alpha_\ell} + \ell \left(t - \frac{1 - e^{-\left(\alpha_i + \alpha_\ell + \alpha_d + \beta_\ell e_p\right)t}}{\alpha_i + \alpha_\ell + \alpha_d + \beta_\ell e_p} \right) \frac{\alpha_i + \alpha_\ell}{\alpha_i + \alpha_\ell + \alpha_d + \beta_\ell e_p} \tag{28}$$

where V_p = volume of supply to the lower soil zone (from the upper soil zone) and the subterranean outflow volume, $V_\ell' = \dfrac{\alpha_\ell}{\alpha_i + \alpha_\ell} V_{i+\ell}'$, from the lower soil zone storage determines the lower soil zone inflow to the next downstream cell. Again, approximating it as constant over the time interval, it is determined by dividing by the length of the time interval. Existing computer code in the LBRM encoding can be applied for upstream flow into the lower soil zone, ℓ, by substituting $\alpha_i + \alpha_\ell$ for α_i in (25) and (26) and correcting each time increment with (27) and (28). The beginning storage in the following time increment is set equal to the ending storage for the present time increment, L_t'. Furthermore, the outflow volume from the cell's lower soil zone storage, V_i', would be used (as V_i) in (8) to compute the basin outflow volume, V_s, which is then used in (13) to compute the corrected basin outflow volume, V_s'.

Flow Network

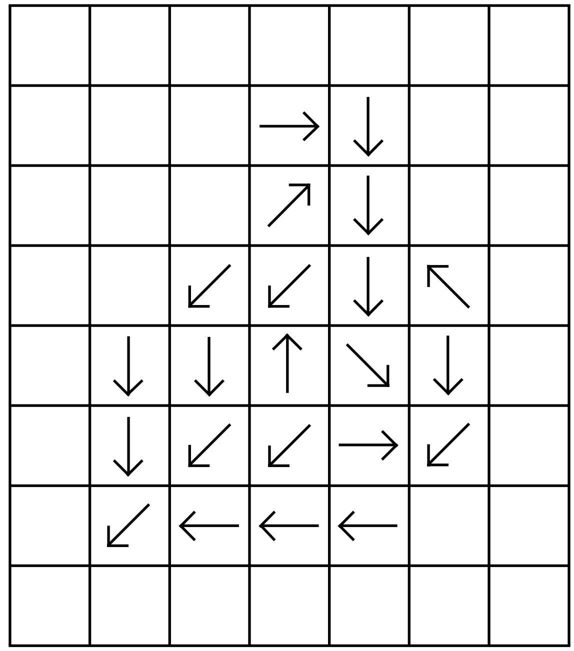

Figure 5. Cell Flow Directions Map.

Consider that a watershed is broken into a group of cells, as in the map of Figure 5. Each cell has flow properties assigned to it and one of eight flow directions, based upon the watershed topography. Each cell has runoff from its surface and subsurface components into its surface channel system, and it has flows from an upstream cell into its surface channel system and subsurface flow system (except for the most-upstream cells). Here the surface and subsurface flow networks are taken as identical. The flow arrows in Figure 5 thus represent the connections between storages at the surface, in the upper soil zone, in the lower soil zone, and in the groundwater zone. There are several general requirements for watershed maps such as Figure 5. One and only one outlet from the watershed must exist. There must be one and only one cell in the watershed whose flow enters an "empty" cell (a cell with no flow designated in the watershed, i.e., a cell that is not within the watershed). All other cells with flows must enter other cells with flows (other cells within the watershed). Furthermore, no "flow loops" may exist, isolating cells from drainage to the outlet. Croley and He (2003) present a micro-hydrology computation-ordering algorithm for application to a well-defined flow network to order cell hydrograph and routing computations. It is useful for checking that one and only one outlet exists and that no flow loops exist.

Information on flow directions can be used to organize runoff and routing computations. The cells with flows entering any given cell can be ascertained by inspection of the flow directions of all surrounding cells. The routing computations are actually programmed with a recursive routine, wherein the routine for determining the flow out of a cell involves successively calling itself to determine the flow out of other cells entering the cell. This implementation requires that the watershed outlet cell be known for the first call to the routine. The recursive routing routine first calls itself for each relevant upstream cell (with flows into the current cell) to determine all inflow hydrographs to the current cell: outflows from upstream surface storages, surface runoff from upstream upper soil zones, interflows from upstream lower surface storages, and groundwater flows from upstream groundwater zones. It then sums all inflows from each zone to determine the total input hydrographs into each of the storage zones of the current cell. Then it does the routing by solving the mass continuity equations for every time interval in the hydrographs; the original model computer code is used with altered parameters in (7), (8), (14), (15), (21), (22), (25), and (26), and corrections are made with (12), (13), (19), (20), (23), (24), (27), and (28). Finally, it assembles the outflow hydrographs, one from each storage in the current cell.

This recursive routing routine enables efficient computations, whereby each cell's outflows (outflow, surface runoff, interflow, and groundwater flow) and upstream flows are determined and routed through the cell only once, with minimum storage of pending hydrographs. (A flow hydrograph out of a cell must be saved as a "pending" inflow hydrograph into the next downstream cell, until all upstream inflows for that next cell are computed; then they are added together to determine the total upstream surface flow into that next cell.)

Summary

GLERL's LBRM continuity equations were modified to allow upstream inflows when the model is applied to a single cell within a watershed. The LBRM is now applied, in both spatial dimensions, to a system of cells comprising a watershed. The inflows to each cell can now consist of outflows from upstream surface storages, surface runoff from upstream upper soil zones, interflow from upstream lower soil zones, and groundwater flows from upstream groundwater zones. The outflows from a cell consist of similar flows from the cell's own moisture storage zones. The modifications to the LBRM were devised in terms of both the original equations (with no upstream/downstream flows) with new parameters (so we can use the same computer code) and new corrector equations to be applied to the original equation solution. LBRM applications to constituent watershed cells are organized in a flow network by identifying the network flow cascade and then automatically arranging the cell computations accordingly. Required characteristics of any flow network map are identified and system checks to guarantee them are designed. These characteristics include the presence of a unique watershed outlet cell and the absence of flow loops within the watershed. A recursive routing routine was devised for subsequently ordering LBRM computations and routing flows throughout the watershed.

Acknowledgments

This is GLERL contribution no. 1296.

References

Croley, T. E., II (1982). "Great Lakes basins runoff modeling." *NOAA Technical Memorandum ERL GLERL-39*, National Technical Information Service, Springfield, Virginia, 22161. 96pp.

Croley, T. E., II (2002). "Large basin runoff model." *Mathematical Models in Watershed Hydrology*, Water Resources Publications, Littleton, Colorado, 717—770.

Croley, T. E., II, and C. He (2003). "Distributed-parameter Large Basin Runoff Model I: model development." *J. Hydrol. Engrg.*, (submitted).

Rainville, E. D., 1964. *Elementary Differential Equations*. Third Edition, MacMillan, New York, New York, 36—39.