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A Method for Parameter Sensitivity Analysis in Differential Equation Models

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A numeric method for analyzing global parameter sensitivity about a fixed point in parameter space for differential equation models is presented. The method is suitable for large-scale, multiresponse systems which may not be in steady state. By using a quadratic model, the relationship between several global response characteristics and parameter perturbations is examined. Sensitivity relationships are defined with both backward elimination regression model selection procedures and eigenvalue-eigenvector analyses. An example of the method is given using an ecosystem model consisting of 14 coupled differential equations.

INTRODUCTION

Differential equation models are useful tools in describing a variety of complex systems, having applications in such fields as economics, medicine [*Jacquez*, 1972], biology [*McNaught* and *Scavia*, 1976], and ecology [Park *et al.*, 1974; *Scavia et al.*, 1976: Thomann *et al.*, 1975]. At present, the primary use of such models is that of prediction. Input (or driving) variables are perturbed and resulting system behavior observed. Model validity is usually defined in terms of predictive capability [*Aigner*, 1972]. Because many of the mathematical relationships used in defining the differential equation system are based on extant scientific principles, knowledge of the system may also be gained by examining the system under internal change, i.e., changes of the parameter values of the mathematical constructs.

Tomovic [1963] defines several sensitivity coefficients based on the sensitivity equation, a differential equation relating the change in response with a change in the parameters. For nonsteady state systems the sensitivity coefficient is a continuous function of time. The usefulness of the method depends on the ability to reformulate the system in analog terms or provide some analytical results for the solution to the sensitivity equation. For many large-scale models this is not feasible. *Steinhorst* and *Gustafson* [1975] determine sensitivity by parameter perturbation and an analysis of variance. Such an approach assumes additive normal error and obscures the continuous relationship between the parameters and the objective criteria. Banning [1974] and *Kleijnen* [1975] discuss a sensitivity method applied to the driving variables of a stochastic simulation model.

This paper will present a method to quantify overall parameter sensitivity relationships with numeric techniques. It will describe an application of empirical model building in a linear regression framework and eigenvalue-eigenvector canonical analysis to evaluate parameter changes and system response. The method is not dependent on either an analytic solution to the differential equations or a formulation to an analog model. It is specifically designed for multiresponse systems.

METHOD

The method is basically to determine some objective criteria for all responses integrated over the primary variable, usually time. Parameters are perturbed from a given parameter set (driving variables being held constant), and new objective criteria determined. The objective criteria are defined as distance measures of the resulting perturbed response from that response obtained from the given parameter set. The relationship between the parameters and the objective criteria is then evaluated using a quadratic model.

The differential equation model may be formulated as

$$\dot{y}_i = f_i(\boldsymbol{\theta}, \mathbf{x}(t), \mathbf{y}, t) \qquad i = 1, 2, \cdots, c \tag{1}$$

where c is the number of responses (compartments), $\dot{y}_i = dy_i/dt$, $\boldsymbol{\theta}$ is the vector of parameter values, y is the vector of response variables, $\mathbf{x}(t)$ is the vector of driving variables, and t is the time. The integrated form of the response will be given as

$$y_i = g_i(\boldsymbol{\theta}, \mathbf{x}(t), \mathbf{y}, t) \qquad i = 1, 2, \cdots, c \tag{2}$$

and the response at the given parameter set as

$$y_i^* = g_i(\boldsymbol{\theta}^*, \mathbf{x}(t), \mathbf{y}, t) \qquad i = 1, 2, \cdots, c \tag{3}$$

In most cases, the analytical solution, $g_i(\ )$, is not known and (2) and (3) must be represented by sets of discrete points over time. The spacing of these discrete points should be such that an adequate representation of the behavior of the responses over time is made. The grid should be the same for all responses or a bias will be introduced into the objective criteria. Usually the grid is easily made because (3) is extensively studied before any sensitivity analysis is done.

The parameters $\boldsymbol{\theta}$ are systematically perturbed from their given values $\boldsymbol{\theta}^*$, and (1) is integrated over a given time frame. Three values are used for each parameter: the given value and $\pm 10\%$ change from the given value. This results in 3^p perturbations, where p is the number of parameters to be examined. The actual percentage perturbation used in the analysis is dependent on the quantity of interest, i.e., the sensitivity of the system to parameter changes. Too large or too small a change may miss important features of the response surface; therefore each system must be dealt with individually. A 10% change has

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TABLE 1. Canonical Analysis for Primary Production Parameter Sensitivity Analysis

	Eigenvalue	*TOPT*	*TMAX*	*Q10*
		Stationary Points		
Objective 1		19.04	34.39	1.76
Objective 2		19.61	34.31	1.83
Given		20.0	35.0	1.9
		Eigenvector		
Objective 1	1954.59	0.1348	−0.0096	0.9908
	−15.35	0.9895	−0.0517	−0.1352
	0.62	−0.0525	−0.9986	−0.0025
Objective 2	62606.99	0.1392	−0.0068	0.9902
	−751.93	0.9903	−0.0028	−0.1392
	15.68	0.0018	−0.9999	−0.0071

proven to be successful for the ecosystem model used in the example.

The formulation of objective criteria that compare the given and perturbed simulations is of prime importance. A simple sum of squares approach is not feasible because observed values of the model responses often differ by orders of magnitude. The following objective criteria were examined:

$$o_{1k} = \sum_{i=1}^{c} \sum_{j=1}^{n} \frac{|y_{ij}{}^* - y_{ijk}|}{y_{ij}{}^*} \tag{4}$$

$$o_{2k} = \sum_{i=1}^{c} \sum_{j=1}^{n} \left(\frac{y_{ij}{}^* - y_{ijk}}{y_{ij}{}^*}\right)^2 \tag{5}$$

where k is the perturbation $k = 1, 2, \cdots, 3^p$, p is the number of parameters examined, c is the number of compartments, n is the number of observations for each response, $y_{ij}{}^*$ is the jth observation for the ith response for the given parameter set, and y_{ijk} is the jth observation for the ith response for the kth perturbed parameter set. Objective criterion 1 (equation (4)) is an estimate of the integral absolute percentage difference, and objective criterion 2 (equation (5)) is an estimate of the integral squared percentage difference. From the form of the objective criteria it is clear that sensitivity is being defined with respect to a given point in the parameter space.

Equation (1) is solved for the perturbed parameter points and the objective criteria calculated. The relationship between the objective criterion and the parameter values is then examined by a quadratic model given by

$$o = \beta_0 + \boldsymbol{\beta}'\boldsymbol{\theta} + \boldsymbol{\theta}'\mathbf{A}\boldsymbol{\theta} \tag{6}$$

$$\boldsymbol{\beta}' = (\beta_1, \beta_2, \cdots, \beta_p)$$

$$\boldsymbol{\theta}' = (\theta_1, \theta_2, \cdots, \theta_p)$$

$$\mathbf{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12}/2 & \cdots & \alpha_{1p}/2 \\ & \alpha_{22} & \ddots & \vdots \\ & & & \vdots \\ & sym & & \alpha_{pp} \end{bmatrix}$$

Such a model is equivalent to a second-order Taylor series approximation to the functionality between the differential model (1) parameters and the objective criterion. The estimation of β_0, $\boldsymbol{\beta}$, and $\mathbf{A}$ of (6) may be done by any standard least-squares program.

The analysis of the fitted surface (6) proceeds along two different but complimentary paths. A canonical analysis is first used to examine the fitted equation (6). By the usual differentiation techniques the stationary points of (6) are given as

$$\boldsymbol{\theta}_0 = -\mathbf{A}^{-1}\boldsymbol{\beta}/2 \tag{7}$$

and the estimated response at this stationary point by

$$\hat{o}_0 = \beta_0 + \boldsymbol{\theta}_0{}'\boldsymbol{\beta}/2 \tag{8}$$

Equation (6) in canonical form is

$$\hat{o} = \hat{o}_0 + \lambda_1 w_1{}^2 + \lambda_2 w_2{}^2 + \cdots + \lambda_k w_k{}^2 \tag{9}$$

where λ_i are the characteristic roots (eigenvalues) of $\mathbf{A}$, $\mathbf{w} = \mathbf{M}(\boldsymbol{\theta} - \boldsymbol{\theta}_0)$, and $\mathbf{M}$ is the matrix whose rows are the normalized eigenvectors associated with the eigenvalues of $\mathbf{A}$. Since $\mathbf{A}$ is real and symmetric, the eigenvalues λ_i are all real, and if $\mathbf{A}$ is of rank p, then p eigenvalues exist.

The characterization of the stationary point (7) is accomplished by inspection of the eigenvalues of (9). Clearly if all eigenvalues are positive, then a move away from $\boldsymbol{\theta}_0$ results in a higher value of the objective criterion and $\boldsymbol{\theta}_0$ then represents a minimum. Similarly, if all eigenvalues are negative, then $\boldsymbol{\theta}_0$

TABLE 2. Canonical Analysis for Decomposition Parameter Sensitivity Analysis

	Eigenvalue	*TOPT*	*TMAX*	*Q10*
		Stationary Points		
Objective 1		22.50	56.47	1.69
Objective 2		22.27	48.56	1.61
Given		25.0	50.0	1.7
		Eigenvector		
Objective 1	3398.08	0.0651	−0.0075	0.9978
	−8.59	0.9707	−0.2311	−0.0651
	0.90	−0.2311	−0.9729	0.0078
Objective 2	−2091.31	−0.1355	0.1961	0.9712
	214.39	−0.9846	0.0825	−0.1541
	103.71	−0.1103	−0.9771	0.1819

TABLE 3. Estimated Models for Primary Production Parameter Sensitivity

Parameter	Coefficient	Standard Error	F to Remove
	Objective 1		
TOPT	−1767.1	270.4	42.7
Q10	−17399.9	2598.9	44.8
$(TOPT)^2$	20.4	6.4	10.1
$(TOPT)(Q10)$	525.9	45.3	134.8
$(Q10)^2$	1924.4	640.7	9.0
	Objective 2		
TOPT	−29959.1	3360.6	79.5
Q10	−550184.4	>10000.0	29.6
$(TOPT)(Q10)$	17457.3	1762.2	98.1
$(Q10)^2$	61568.1	24922.0	6.1

These are estimated models for (6) using the stepwise backward elimination technique. The F to remove value was used to test significance of the parameters in the model.

represents a maximum. Although the stationary points (7) should represent a minimum, in practice the eigenvalues will often be mixed in sign. Such a mixture is usually indicative of a saddle point. The first qualitative measure concerning the fit of (6) is that even though a saddle point may be obtained, the stationary point (7) should lie close to the given parameter set. If not, then the fitted equation is of little value in examining sensitivity.

If the eigenvalues are now ranked from largest to smallest, the influence of the w_i and hence $\boldsymbol{\theta}$ can clearly be seen. The largest eigenvalues have the greatest effect on the objective criteria, and the θ_i or combination of θ_i having the greatest effect on the objective function can be determined by inspection of the associated eigenvectors.

The examination of the eigenvalues and eigenvectors of A give a qualitative indication of the sensitivities of the parameters $\boldsymbol{\theta}$. In the example to be discussed here the sensitivities of the parameters are clear because the eigenvalues differ by orders of magnitude. In other cases the determination is not so easy, and a statistical approach must be used.

If we assume that deviations from the model (6) are independent and normally distributed with constant variance, then model fitting in a linear regression framework may be used to determine those parameters $\boldsymbol{\theta}$ that contribute significantly to the fitting of the objective criterion and are hence the most sensitive in equation (1). A backward elimination procedure is well suited for this purpose. In this procedure, variables (i.e., elements of $\boldsymbol{\beta}$ and $\mathbf{A}$) are successively removed from the model (5) according to some preselected probability level. The parameters $\boldsymbol{\theta}$ remaining in the final reduced model are those that significantly explain variations in the objective criterion and may be judged as the most sensitive parameters.

TABLE 4. Estimated Models for Decomposition Parameter Sensitivity

Parameter	Coefficient	Standard Error	F to Remove
TOPT	−716.1	176.4	16.5
Q10	−10670.3	2594.7	16.9
$(TOPT)(Q10)$	442.6	103.4	18.3

These are estimated models for objective 1 and (6) using the stepwise backward elimination technique. F to remove value was used to test significance of the parameters in the model. The model fit by objective 2 is not satisfactory with the full model having $F(9, 17) = 1.58$. There were no significant variables ($P > 0.05$).

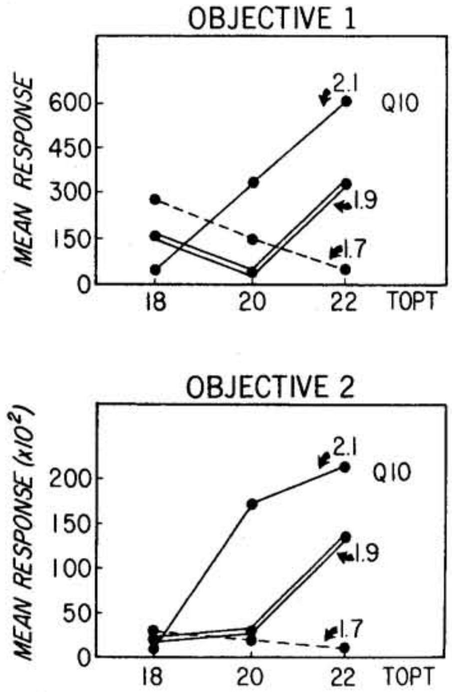

Fig. 1. Primary production parameters for interaction plot.

EXAMPLE

The method developed here was applied to a lake ecosystem model [*Scavia et al.*, 1974; *Park et al.*, 1974] describing the open-water zone of Lake George, New York. This model

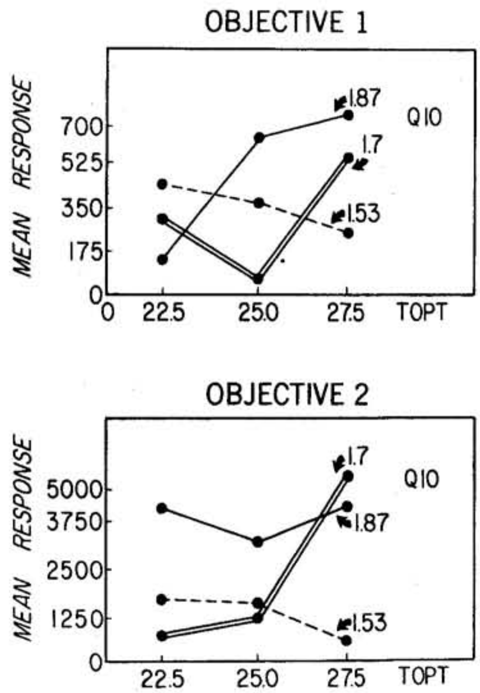

Fig. 2. Decomposition parameters for interaction plot.

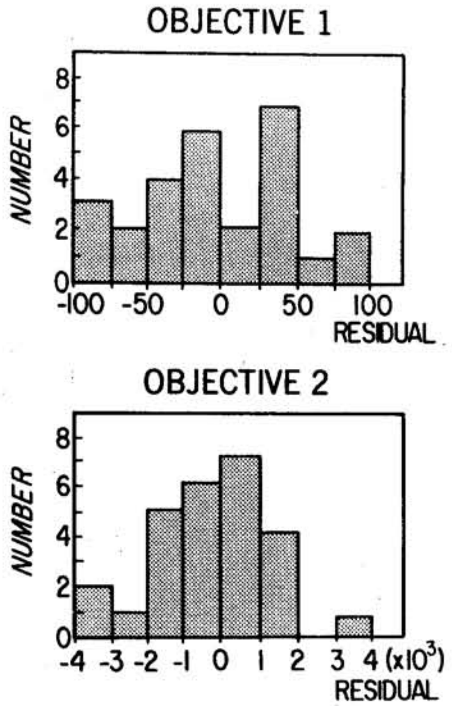

Fig. 3. Primary production parameters for residual histogram.

consists of 14 coupled nonlinear differential equations with over 150 parameters. To determine the utility of the method, two small parameter sets were analyzed independently. One set describes the temperature effect on primary production and the other the temperature effect on decomposition. In both sets the temperature parameters are the upper lethal temperature *TMAX*, the optimal temperature *TOPT*, and the slope of the suboptimal process rate curve *Q*10. These parameters were chosen because of a detailed study of this temperature construct [*Scavia and Park*, 1976]. Equation (1) was integrated over a 1-yr period with a grid size of 5 days. The given parameter values about which sensitivity is measured were known to produce results consistent with ecological theory and available data [*Scavia and Park*, 1976].

The stationary points obtained from (7) all lie close to the given values (Tables 1 and 2). The associated eigenvalues are mixed in sign indicating that the stationary points are not unique minimums. With the exception of objective criterion 2 with decomposition parameters, the largest eigenvalue differs by several orders of magnitude from the others (Tables 1 and 2). By examination of the eigenvector transformation to original coordinates, it is seen that the effect of *TMAX* is an order of magnitude smaller than *TOPT* or *Q*10. *TOPT* and *Q*10 are similar in the magnitude of their effects on the objective criteria. Also the crossproduct of *Q*10 and *TOPT* is seen to have a large effect.

Under the appropriate error assumptions the reduced model given by backward elimination confirms the results from the canonical analysis (Tables 3 and 4). The fit for objective criterion 2 for decomposition is not statistically significant. The crossproduct of *Q*10 and *TOPT*, representing interactions, is significant. The effect of *Q*10 and *TOPT* is seen to be compensatory (Figures 1 and 2). This compensatory effect is supported by theoretical considerations [*Scavia and Park*, 1976]. Examination of the residuals (Figures 3 and 4) reveals no serious departure from normality although objective criterion 1 with decomposition may represent a uniform distribution. This uniform distribution will not greatly affect the *F* tests for it is nonsymmetric distributions which have the greatest effect on the stated probability level of the *F* test.

DISCUSSION

The proposed method obtains global parameter sensitivity results consistent with known relationships. It should prove useful in examining model constructs and parameters where theoretical conclusions are not available. For the parameters of the example, the objective criteria of normalized absolute deviation provide a better fit to a quadratic model than a normalized sum of squares. Both criteria should be calculated. The cost of these calculations is minimal in comparison with the amount of computer time necessary to integrate the differential equations, and the degree of consistency between results provides a check of the analysis.

In the example provided for demonstration purposes for $p = 3$, only $3^3 = 27$ different parameter sets needed to be examined. In this instance, the computation was not excessive for examining all combinations. However, for $p > 3$, the amount of computing would build up very rapidly. When this is the case, it will be wise to look at only a subset of the possible 3^p combinations. One method would be to analyze in previously defined subsets like those in the example. Although in some models it is easy to define subsets, such an approach precludes sensitivity comparisons between the subsets. Another procedure would be to take a balanced fraction, 3^{-q}, so that only 3^{p-q} combinations would need to be examined. Assistance in choosing such subsets for various values of p and q is provided by tables [*National Bureau of Standards*, 1959]. Fairly substantial initial fractionalizing would be recommended, with sequential augmentation of subsequent blocks of additional fractions if more precision is needed.

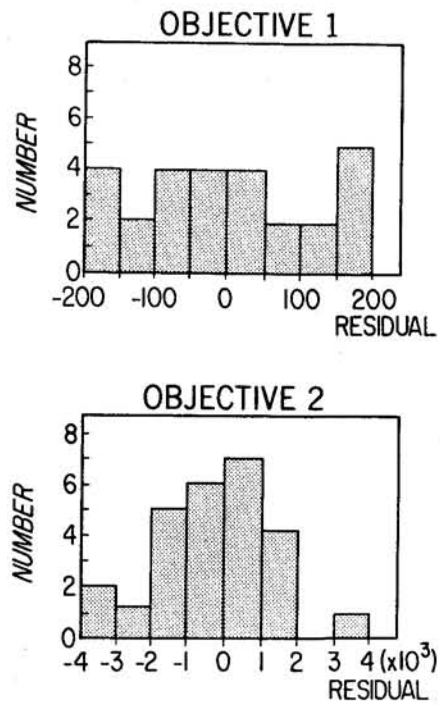

Fig. 4. Decomposition parameters for residual histogram.

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REFERENCES

Aigner, D. J., A note on verification of computer simulation models, *Manage. Sci., 18*, 615–619, 1972.

Blanning, R. W., The sources and uses of sensitivity information, *Interfaces, 4*(4), 32, 1974.

Jacquez, J. A., *Compartmental Analysis in Biology and Medicine*, Elsevier, New York, 1972.

Kleijnen, J. P. C., A comment on Blanning's 'Metamodel for sensitivity analysis: The regression model in simulation,' *Interfaces, 5*(3), 21, 1975.

McNaught, D. C., and D. Scavia, Application of a model of zooplankton composition to problems of fish introduction in the Great Lakes, in *Modeling Biochemical Processes in Aquatic Systems*, edited by R. P. Canale, pp. 281–304, Ann Arbor Science, Ann Arbor, Mich., 1976.

National Bureau of Standards, Fractional factorial experimental designs for factors at three levels, *Appl. Math. Ser., 54*, Washington, D. C., 1959.

Park, R. A., et al., A generalized model for simulating lake ecosystems, *Simulation, 23*(2), 33–50, 1974.

Scavia, D., and R. A. Park, Documentation of selected constructs and parameter values in the aquatic model CLEANER, *Ecol. Modelling, 2*(1), 33–58, 1976.

Scavia, D., B. J. Eadie, and A. Robertson, An ecological model for Lake Ontario—Formulation, calibration, and preliminary evaluation, *NOAA Tech. Rep., ERL 371-GLERL12*, Boulder, Colo., 1976.

Scavia, D., et al., Documentation of CLEANX: A generalized model for simulating the open-water ecosystems of lakes, *Simulation, 23*(2), 51–56, 1974.

Steinhorst, R. K., and F. P. Gustafson, Parameter estimation in nonlinear simulation models, paper presented to the annual meeting, Amer. Statist. Ass., Atlanta, Ga., Aug. 1975.

Thomann, R. V., D. M. DiToro, R. P. Winfield, and D. J. O'Connor, Mathematical modeling of phytoplankton in Lake Ontario, 1, Model development and verification, *EPA Rep. EPA-660/3-75-005*, Environ. Protect. Agency, Corvallis, Ore., 1975.

Tomovic, R., *Sensitivity Analysis of Dynamic Systems*, McGraw-Hill, New York, 1963.

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